Application of linear response theory to magnetotransport properties of dense plasmas

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Linear response theory, as developed within the Zubarev formalism, is a quantum statistical approach for describing systems out of but close to equilibrium, which has been successfully applied to a wide variety of plasmas in an external electric field and/or containing a temperature gradient. We present here an extension of linear response theory to include the effects of an external magnetic field. General expressions for the complete set of relevant transport properties are given. In particular, the Hall effect and the influence of a magnetic field on the dc electrical conductivity are discussed. Low-density limits including electron-electron scattering are presented as well as results for arbitrary degeneracy.

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I. INTRODUCTION

The ability to transmit electrical currents has long been seen as a fundamental property of a plasma and the measurement of the electrical conductivity can provide important information on the ionizational stage. Hall used crossed electric and magnetic fields to demonstrate electrons as mobile charge carriers in solid metals [1]. This is still applied to semiconductors in order to determine both the type and density of charge carriers. Recent measurements of the Hall coefficient in dense plasmas motivate investigation into whether similar information can be obtained from plasma systems [2,3].

A widely applied approach to electron transport in magnetized plasmas is the relaxation-time approximation (RTA), in particular as described by Lee and More [4]. The relaxation time is constructed assuming that the electron collision frequency can be described by the sum of the independent collision frequencies of electrons with each particle species in the system. The electron-electron interactions, however, cannot be included directly within the RTA, as these interactions conserve momentum within the electronic subsystem. Instead, the influence of these interactions is often approximated using an interpolating function between the known limiting cases at high and low electron densities [5–7].

Solving the Fokker-Planck equation [8] also allows the calculation of plasma transport properties in magnetic fields. Brown and Haines [9,10] extended this approach to focus on the influence of electron-electron collisions and the effect of partial degeneracy. The electron-electron contribution to the collision frequency is based on the electron-ion collisions, not taking into account higher orders of scattering cross sections. The electron-ion contribution to the collision frequency is determined via the Coulomb logarithm, for which the cutoffs, applied in the degenerate limit according to the Fermi energy, do not reproduce scattering cross sections for strong collisions as calculated using the T-matrix approach [11]. Note that the approximation of the collision frequency by an analytic expression for the Coulomb logarithms is valid only in the low-density limit.

Linear response theory (LRT) as proposed by Zubarev [12,13] provides a description of transport properties in terms of force-force correlation functions (FFCFs) which are readily calculated [13–15] within perturbation theory. The interactions between the electrons and all other species are described within the FFCFs, enabling a complete description of partially ionized plasmas [11,16–18]. Electron degeneracy is taken into account and the T-matrix approach is applied where appropriate [19].

In this work, we present modifications necessary to the existing form of LRT to include the effects of an external magnetic field [20]. We begin in Sec. II by giving a brief overview of the phenomenological relations and corresponding transport coefficients relevant to a system in external electric and magnetic fields and containing a temperature gradient. In Sec. III, we derive the formal expression in LRT for the complete set of thermoelectric and magnetoelectric transport coefficients. In Secs. IV and V, the results for fully and partially ionized plasmas, respectively, are discussed. Finally, the possibility of using measurements of transport coefficients as a diagnostic tool for determining the free-electron density of a plasma is investigated. Calculations using argon data as an example are presented. Within this work, two dimensionless parameters are used to characterize the plasma state

$$\Gamma = \frac{e^2\beta}{4\pi\varepsilon_0}\left(\frac{4\pi n_e}{3}\right)^{1/3}, \quad \Theta = \frac{2m_e}{\beta\hbar^2}(3\pi^2 n_e)^{-2/3}, \qquad (1)$$

where $\beta=1/(k_BT)$ is the inverse temperature and m_e and n_e are the electron mass and free-electron density, respectively. The coupling parameter Γ is the ratio of the average potential energy to the temperature (in units of energy) and the degeneracy parameter Θ is the ratio of the temperature to the Fermi energy.

II. TRANSPORT COEFFICIENTS IN CROSSED FIELDS

A system with static external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields and containing a uniform temperature gradient ∇T is considered. We present phenomenological relations for the electrical $\boldsymbol{j}$ and energy $\boldsymbol{Q}$ current densities

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$$\boldsymbol{j} = \hat{\sigma}[\boldsymbol{E} - \hat{S}\,\boldsymbol{\nabla} T + R_H(\boldsymbol{j}\times\boldsymbol{B}) + N(\boldsymbol{\nabla} T\times\boldsymbol{B})], \tag{2}$$

$$\boldsymbol{Q} = T\hat{S}\boldsymbol{j} - \hat{\kappa}\,\boldsymbol{\nabla} T - NT(\boldsymbol{j}\times\boldsymbol{B}) - L(\boldsymbol{\nabla} T\times\boldsymbol{B}), \tag{3}$$

which define the transport properties in the same way as Lee and More [4,21] or Brown and Haines [10]. The transport coefficients $\hat{\sigma}$, $\hat{\kappa}$, and $\hat{S}$ are tensors of electrical conductivity, heat conductivity, and thermopower, respectively. Assuming isotropy with respect to $\boldsymbol{E}$, these coefficients become anisotropic in the presence of a magnetic field and are given as diagonal tensors. R_H, N, and L are the Hall, Nernst, and Leduc-Righi coefficients, respectively. They are only relevant in the plane perpendicular to $\boldsymbol{B}$ and are isotropic in this plane. An alternative description of the transport coefficients used by Haines *et al.* [8,10,22] is the introduction of nondiagonal elements in the resistivity tensor and the tensors for heat conductivity and thermoelectric coefficient, respectively. However, here we use the phenomenological relations (2) and (3) to make a comparison to the experiment. The direction of $\boldsymbol{B}=B\hat{e}_z$ defines the reference axis in the system and currents are distinguished as parallel $\parallel$ or perpendicular $\perp$ to this z direction. Then, the electrical conductivity tensor is

$$\hat{\sigma} = \begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}, \tag{4}$$

with similar definitions for $\hat{\kappa}$ and $\hat{S}$. $\boldsymbol{E}$ and $\boldsymbol{\nabla}T$ are also given by components parallel and perpendicular to $\boldsymbol{B}$. Note that the perpendicular components need not be resolved into x and y components. Resolving Eqs. (2) and (3) into components, we obtain in the direction parallel to $\boldsymbol{B}$,

$$\boldsymbol{j}_\parallel = \sigma_\parallel(\boldsymbol{E}_\parallel - S_\parallel\,\boldsymbol{\nabla} T_\parallel), \tag{5}$$

$$\boldsymbol{Q}_\parallel = TS_\parallel \boldsymbol{j}_\parallel - K_\parallel\,\boldsymbol{\nabla} T_\parallel, \tag{6}$$

and in the plane perpendicular to $\boldsymbol{B}$,

$$\boldsymbol{j}_\perp/\eta = \sigma_\perp \boldsymbol{E}_\perp - (\sigma_\perp S_\perp + \sigma_\perp^2 R_H N B^2)\,\boldsymbol{\nabla} T_\perp + [\sigma_\perp^2 R_H \boldsymbol{E} + (\sigma_\perp N - \sigma_\perp^2 R_H S_\perp)\,\boldsymbol{\nabla} T_\perp]\times\boldsymbol{B}, \tag{7}$$

$$\boldsymbol{Q}_\perp = TS_\perp \boldsymbol{j}_\perp - \kappa_\perp\,\boldsymbol{\nabla} T_\perp - (NT\boldsymbol{j}_\perp - L\,\boldsymbol{\nabla} T_\perp)\times\boldsymbol{B}, \tag{8}$$

$$\eta = (1 + \sigma_\perp^2 R_H^2 B^2)^{-1}. \tag{9}$$

It should be noted that in the following, we assume a linear response of the system to $\boldsymbol{E}$ and $\boldsymbol{\nabla}T$. Nonlinear behavior is observed only in very strong fields, such as intense laser pulses, and is not relevant to the systems considered in this work.

III. LINEAR RESPONSE THEORY

Following the LRT in the formulation of Zubarev, a set of suitably chosen relevant nonequilibrium observables is used to derive the statistical operator [11,13]. In previous works, the generalized linear momenta,

$$\hat{\boldsymbol{P}}_n = \sum_{\boldsymbol{k}} \hbar\boldsymbol{k}(\beta\epsilon_k)^n \hat{a}_e^\dagger(\boldsymbol{k})\hat{a}_e(\boldsymbol{k}), \tag{10}$$

were chosen as the relevant nonequilibrium observables. The momentum of each electron is weighted by a varying power n of its classical kinetic energy ϵ_k and $\hat{a}_e^\dagger(\boldsymbol{k})\hat{a}_e(\boldsymbol{k})$ is the number operator for electrons with wave vector $\boldsymbol{k}$. In order to ensure that the Lorentz force is consistently included when considering an external magnetic field $\boldsymbol{B}$, it is more convenient to instead of $\hat{\boldsymbol{P}}_n$ use the generalized canonical momenta $\dot{\hat{\boldsymbol{R}}}_n$ as observables. This is achieved by using the generalized position operator

$$\hat{\boldsymbol{R}}_n = i\sum_{\boldsymbol{k}\boldsymbol{k}'} (\beta\epsilon_{k'})^n \delta(\boldsymbol{k}-\boldsymbol{k}')\frac{\partial}{\partial\boldsymbol{k}}\hat{a}_e^\dagger(\boldsymbol{k}')\hat{a}_e(\boldsymbol{k}) \tag{11}$$

and taking the time derivative $\dot{\hat{\boldsymbol{R}}}_n=\frac{i}{\hbar}[\hat{H},\hat{\boldsymbol{R}}_n]$. The Hamiltonian $\hat{H}=\hat{H}_s+\hat{H}_1+\hat{H}_2$ contains the equilibrium term $\hat{H}_s=\sum_{\boldsymbol{k}}\epsilon_k\hat{a}_e^\dagger(\boldsymbol{k})\hat{a}_e(\boldsymbol{k})+\sum_c\hat{V}^{ec}$ as well as nonequilibrium contributions from each of the external perturbations

$$\hat{H}_1 = e\boldsymbol{E}\cdot\hat{\boldsymbol{R}}_0, \tag{12}$$

$$\hat{H}_2 = -\frac{e}{2}(\hat{\boldsymbol{R}}_0\times\dot{\hat{\boldsymbol{R}}}_0)\cdot\boldsymbol{B}. \tag{13}$$

Performing time derivatives, we obtain the generalized canonical momenta and forces

$$m_e\dot{\hat{\boldsymbol{R}}}_n = \hat{\boldsymbol{P}}_n - \frac{e}{2}(\hat{\boldsymbol{R}}_n\times\boldsymbol{B}), \tag{14}$$

$$m_e\ddot{\hat{\boldsymbol{R}}}_n = \sum_c \hat{\boldsymbol{F}}_n^{ec} - e\boldsymbol{E}\sum_{\boldsymbol{k}}(\beta\epsilon_k)^n\hat{a}_e^\dagger(k)\hat{a}_e(k) - e(\dot{\hat{\boldsymbol{R}}}_n\times\boldsymbol{B}), \tag{15}$$

where $\hat{\boldsymbol{F}}_n^{ec}=\frac{i}{\hbar}[\hat{V}^{ec},\hat{\boldsymbol{P}}_n]$ represents the force due to interactions between the electrons and species c.

Averages of the relevant nonequilibrium observables are assumed to be given exactly by averaging with $\hat{\varrho}_{\rm rel}$, leading to the self-consistency condition

$$\langle\dot{\hat{\boldsymbol{R}}}_n\rangle = \mathrm{Tr}\{\hat{\varrho}\cdot\dot{\hat{\boldsymbol{R}}}_n\} \equiv \mathrm{Tr}\{\hat{\varrho}_{\rm rel}\cdot\dot{\hat{\boldsymbol{R}}}_n\}, \tag{16}$$

where $\hat{\varrho}$ is the full nonequilibrium statistical operator [13]. A significant advantage of this choice of relevant observables is that they can be used directly to calculate current densities

$$\boldsymbol{j} = -\frac{e}{\Omega_0}\langle\dot{\hat{\boldsymbol{R}}}_0\rangle, \quad \boldsymbol{Q} = \frac{k_BT}{\Omega_0}\langle\dot{\hat{\boldsymbol{R}}}_1\rangle + \frac{h}{e}\boldsymbol{j}, \tag{17}$$

where Ω_0 is the normalization volume and $h=\mu_e-T(\frac{\partial\mu_e}{\partial T})_{V,N}$ is the specific enthalpy.

The electric field and the temperature gradient are the primary nonequilibrium perturbations driving electrical currents in the system. Since the magnetic field creates only secondary currents via the Lorentz force, we should expect $\boldsymbol{B}$ to appear only as a cross product with $\boldsymbol{E}$ or ∇T. Importantly then, linearization of $\hat{\varrho}_{\rm rel}$ is performed while allowing the size of the magnetic field to remain arbitrary. The electrical and energy current densities parallel and perpendicular to the magnetic field are found and compared to Eqs. (5)–(9). Microscopically derived expressions for the transport coefficients are extracted. We present results for the transport coefficients using the same notation as Lee and More [4].

Perpendicular to $\boldsymbol{B}$, the electrical conductivity, thermopower, and heat conductivity are given by

$$\sigma_\perp = e^2 \frac{\tilde{K}_{01}}{\eta}, \tag{18}$$

$$S_\perp = \frac{1}{eT}\left(h - \frac{\tilde{K}_{11}}{\tilde{K}_{01}}\gamma\eta\right), \tag{19}$$

$$\kappa_\perp = \frac{1}{T}\left[\tilde{K}_{21}\left(1 + \omega_e^2 \frac{\tilde{K}_{12}^2}{\tilde{K}_{01}\tilde{K}_{21}}\right) - \frac{\tilde{K}_{11}^2}{\tilde{K}_{01}}\gamma^2\eta\right], \tag{20}$$

with

$$\eta = \left(1 + \omega_e^2 \frac{\tilde{K}_{02}^2}{\tilde{K}_{01}^2}\right)^{-1}, \quad \gamma = 1 + \omega_e^2 \frac{\tilde{K}_{02}\tilde{K}_{12}}{\tilde{K}_{01}\tilde{K}_{11}} \tag{21}$$

and $\omega_e = eB/m_e$ is the cyclotron frequency. The temperature, density, and magnetic field dependent functions $\tilde{K}_{ij}$ are given in determinant form

$$\tilde{K}_{01} = -\frac{\beta}{\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_0 \\ \bar{N}_0 & \boldsymbol{D}\end{vmatrix},$$

$$\tilde{K}_{11} = -\frac{1}{\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_0 \\ \bar{N}_1 & \boldsymbol{D}\end{vmatrix},$$

$$\tilde{K}_{21} = -\frac{1}{\beta\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_1 \\ \bar{N}_1 & \boldsymbol{D}\end{vmatrix}, \tag{22}$$

where $N_m = \{N_{0m}N_{1m}\ldots\}$ and $\bar{N}_n = \{N_{n0}N_{n1}\ldots\}^T$ are row and column vectors, respectively.

These expressions are found to contain Kubo products $(\hat{X};\hat{Y})$ and correlation functions $\langle\hat{X};\hat{Y}\rangle$, which are defined for two observables $\hat{X}$ and $\hat{Y}$ by

$$(\hat{X};\hat{Y}) = \frac{1}{\beta}\int_0^\beta d\tau\, \mathrm{Tr}\{\varrho_0 e^{\tau\hat{H}_s}\hat{X}e^{-\tau\hat{H}_s}\hat{Y}\}, \tag{23}$$

$$\langle\hat{X};\hat{Y}\rangle = \lim_{\varepsilon\to 0}\int_{-\infty}^0 dt' e^{\varepsilon t'}(e^{i\hat{H}_s t'/\hbar}\hat{X}e^{-i\hat{H}_s t'/\hbar};\hat{Y}). \tag{24}$$

The following Kubo products and correlation functions must be calculated [20]

$$\boldsymbol{N} = \frac{1}{m_e}(\hat{\boldsymbol{P}};\hat{\boldsymbol{P}}) \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{d} + \omega_e^2\boldsymbol{A}, \tag{25}$$

$$\boldsymbol{A} = \langle\hat{\boldsymbol{P}};\hat{\boldsymbol{P}}\rangle \quad \text{and} \quad \boldsymbol{d} = \langle\hat{\boldsymbol{F}};\hat{\boldsymbol{F}}\rangle, \tag{26}$$

which are given in matrix form such that, for example, $\langle\hat{\boldsymbol{F}};\hat{\boldsymbol{F}}\rangle$ is the matrix with elements $\langle\Sigma_c\hat{\boldsymbol{F}}_n^{ec};\Sigma_c\hat{\boldsymbol{F}}_m^{ec}\rangle$. The elements of $\boldsymbol{d}$ are known as FFCF which can be separated into contributions from electron-ion (ei), electron-electron (ee), and electron-atom (ea) collisions such that $d_{nm} = d_{nm}^{\mathrm{ei}} + d_{nm}^{\mathrm{ee}} + d_{nm}^{\mathrm{ea}}$. Detailed descriptions for calculating $\boldsymbol{N}$ and $\boldsymbol{d}$ can be found in Refs. [11,17,20].

Note in particular the additional term in the FFCFs (25) due to the matrix $\boldsymbol{A}$. The momentum autocorrelation functions in $\boldsymbol{A}$ are divergent when calculated directly. Notably, $A_{00} = \langle\boldsymbol{P}_0;\boldsymbol{P}_0\rangle$ is the current-current correlation function obtained by Kubo to describe the electrical conductivity [23]. The equivalence between Kubo's description of the electrical conductivity through a current-current correlation function and the Zubarev result containing FFCFs is shown by the relation

$$\langle\boldsymbol{P};\boldsymbol{P}\rangle = (\boldsymbol{P};\boldsymbol{P})\frac{1}{\langle\boldsymbol{F};\boldsymbol{F}\rangle}(\boldsymbol{P};\boldsymbol{P}), \tag{27}$$

where the expression on the right provides a convergent description of $\boldsymbol{A}$ when all Kubo products and correlation functions are treated in matrix form [13]. Using Cramer's rule, elements of the matrix $\boldsymbol{A}$ can thus be written in the determinant form

$$A_{nm} = -\frac{m_e^2}{|\boldsymbol{d}|}\begin{vmatrix} 0 & N_m \\ \bar{N}_n & \boldsymbol{d}\end{vmatrix}. \tag{28}$$

The convergence of the momentum autocorrelation function when calculated perturbatively in this way provides a consistent description of transport properties, while allowing the inclusion of all interactions, in particular the electron-electron correlations.

In order to give a complete set of transport coefficients, we present the expressions for the Hall coefficient, Nernst coefficient, and Leduc-Righi coefficient

$$R_H = -\frac{1}{em_e}\frac{\tilde{K}_{02}}{\tilde{K}_{01}^2}\eta \tag{29}$$

$$N = -\frac{1}{m_e T}\frac{\tilde{K}_{12}}{\tilde{K}_{01}}\left(1 - \frac{\tilde{K}_{02}\tilde{K}_{11}}{\tilde{K}_{01}\tilde{K}_{12}}\right)\eta, \tag{30}$$

$$L = -\frac{e}{m_e T}\left[\tilde{K}_{22} - \frac{\tilde{K}_{11}\tilde{K}_{12}}{\tilde{K}_{01}} - \frac{\tilde{K}_{11}\tilde{K}_{12}}{\tilde{K}_{01}}\left(1 - \frac{\tilde{K}_{02}\tilde{K}_{11}}{\tilde{K}_{01}\tilde{K}_{12}}\right)\gamma\eta\right], \tag{31}$$

respectively, with

$$\tilde{K}_{02} = -\frac{\beta}{\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_0 \\ \bar{A}_0 & \boldsymbol{D}\end{vmatrix},$$

$$\tilde{K}_{12} = -\frac{1}{\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_0 \\ \bar{A}_1 & \boldsymbol{D}\end{vmatrix},$$

TABLE I. Dimensionless transport coefficients f, a, and L [see Eq. (34)] in the low-density limit for fully ionized plasma (ei+ee) and Lorentz plasma (ei). Ten-moment approach within LRT is compared to RTA and Spitzer.

	f		a		L	
$B\to 0$	ei	ei+ee	ei	ei+ee	ei	ei+ee
LRT	1.0159	0.5910	1.5000	0.7036	4.0000	1.6201
RTA	1.0159		1.5		4	
Spitzer	1.0159	0.5908	1.5	0.7033	4	1.6218

$$\tilde{K}_{22} = -\frac{1}{\beta\Omega_0|\boldsymbol{D}|}\begin{vmatrix} 0 & N_1 \\ \bar{A}_1 & \boldsymbol{D} \end{vmatrix}, \tag{32}$$

where $A_m=\{A_{0m}A_{1m}\ldots\}$ and $\bar{A}_n=\{A_{n0}A_{n1}\ldots\}^T$ are row and column vectors, respectively.

The magnetic field terms in the results above are of second order with respect to ω_e, which is found in the RTA [4] as well. Expressions of this form within LRT have not previously been published and provide a similar structure to the transport coefficients as the RTA.

Comparing the results above to RTA expressions for the electrical conductivity $\sigma=e^2 n_e \tau/m_e$, the anisotropic relaxation time τ can be described within LRT by

$$\tau_\perp = \frac{m_e}{n_e}\frac{\tilde{K}_{01}}{\eta}. \tag{33}$$

The factor $\omega_e\tau$ gives the ratio of the cyclotron frequency to the collision frequency and can be used as an indication of the relative strength of the magnetic field. Plasmas with $\omega_e\tau \gg 1$ are said to be magnetized.

The transport coefficients $\sigma_\parallel$, $S_\parallel$, and $\kappa_\parallel$ in the direction parallel to $\boldsymbol{B}$ are obtained from the above expression if the cyclotron frequency is set to zero. Note that results from this procedure are identical to the descriptions presented in previous works in which an external magnetic field was not considered (see, e.g., [14,15,17]). This is expected since the Lorentz force does not affect motion in the direction parallel to $\boldsymbol{B}$.

IV. FULLY IONIZED PLASMAS

In fully ionized plasmas, contributions to the FFCFs come only from ei and ee collisions, such that $d_{nm}=d_{nm}^{\mathrm{ei}}+d_{nm}^{\mathrm{ee}}$. A Lorentz plasma can also be modeled by considering only the ei interactions, e.g., $d_{nm}=d_{nm}^{\mathrm{ei}}$.

The limit of low density and high temperature is now considered for the comparison of theoretical approaches since analytical results can be obtained within both the RTA and LRT [4,14,15]. We define the dimensionless coefficients f, a, and L for the electrical conductivity, thermopower, and the Lorentz number, respectively [14],

$$f = \sigma \ln\Lambda \frac{\sqrt{m_e\beta^3}}{e^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2,$$

$$a = -\frac{e}{k_B}S,$$

$$L = \frac{e^2}{k_B^2 T}\frac{\kappa}{\sigma}, \tag{34}$$

where $\ln\Lambda$ is the low-density Coulomb logarithm for a plasma with singly charged ions, given in the classical limit ($\Gamma\Theta^2 \gg 1$) by $\ln\Lambda=\frac{1}{2}\ln(\frac{\Theta}{\Gamma})$ [14]. For more discussion on Coulomb logarithms, see Adams *et al.* [7,11,12,14] and Brown and Haines [9]. The accuracy of LRT increases as the number of moments is increased if the FFCFs are evaluated using perturbation theory. Numerical calculations have previously been truncated to three or five moments [14,18] while ten are used in this work which ensures convergence to better than 1% (see Table I). Excellent convergence to the results obtained by Spitzer and Härm [24] and within the RTA [4] in the low magnetic field limit is achieved. In addition, we also use

$$r_H = -e n_e R_H,$$

$$N^* = \frac{e}{k_B}\frac{N}{\sigma R_H},$$

$$L^* = \frac{e^2}{k_B^2 T}\frac{L}{\sigma^2 R_H}, \tag{35}$$

as the dimensionless Hall, Nernst, and Leduc-Righi *factors*, respectively.

In the classical low-density limit, the transport coefficients can be written as a fraction of polynomials in terms of the factor $X=\omega_e^2\tau_0^2$, where $\tau_0=m_e N_{00}/d_{00}$ is the one-moment approximation of the relaxation time $\tau_\perp$ (33). For example, the electrical conductivity $\sigma_\perp$ given by Eq. (18) can be written as

$$\sigma_\perp = \frac{c_{00}+c_{01}X+c_{02}X^2+\cdots}{c_{10}+c_{11}X+c_{12}X^2+\cdots}, \tag{36}$$

where c_{ij} are coefficients. Lee and More considered the limit $X\to 0$ by taking first the small field approximation $\boldsymbol{B}\to 0$ and then the low-density limit. In this case, the transport coefficients f, a, and L become isotropic and results in Table I are valid in all directions.

Results for the coefficients describing magnetic field effects in the limit $X\to 0$ are shown in Table II. As the number

TABLE II. Dimensionless transport coefficients r_H, N^*, and L^* [see Eq. (35)] in the low-density limit for fully ionized plasma (ei+ee) and Lorentz plasma (ei).

$B\to 0$	r_H		N^*		L^*	
$\{\dot{\boldsymbol{R}}_n\}$	ei	ei+ee	ei	ei+ee	ei	ei+ee
0	1	1	0	0	0	0
0,1	1.5325	1.2586	0.1782	0.1772	0.0914	0.1529
0,1,2	1.9786	1.2068	1.3602	0.3442	4.6797	1.2530
0,1,2,3	1.9343	1.2077	1.4807	0.3756	8.0000	1.2626
0,1,2,3,4	1.9333	1.2036	1.4988	0.3694	7.7526	1.2881
⋮						
0,…,9	1.9328	**1.1994**	1.5000	**0.3719**	7.7500	**1.2894**
RTA	1.9328		1.5		7.75	

of moments is increased, convergence is achieved. In the case of a Lorentz plasma, we find agreement of LRT to the results of the RTA. Note that Spitzer did not consider magnetic field effects. New limits (bold numbers in the table) are obtained for the magnetotransport coefficients when taking into account ee collisions. Similar to the results for f, a, and L, the inclusion of ee interactions causes reductions in r_H, N^*, and L^* from the values for a Lorentz plasma. Of particular interest is the Hall factor r_H since it is the most readily measured of the electromagnetic transport coefficients. The *standard* value of $r_H=1$ can be determined analytically in both the low-density and degenerate limits. This value is indeed observed in measurements of the Hall effect in solid metals. The RTA predicts deviation from the standard value to $r_H=1.93$ at low density. Including ee interactions, the LRT predicts a smaller limit of $r_H=1.20$. A description of the Hall coefficient within the Zubarev formalism by Schiller *et al.* [25] did not produce any deviation from the standard value as the statistical operator was linearized with respect to the magnetic field, in contrast to the procedure presented above in Sec. III (see also French [26]).

In order to correctly predict the behavior of a low-density plasma in the presence of a magnetic field however, we must assume that the field strength remains finite as $n_e\to 0$. This is because cyclotron motion will begin to dominate as the number of collisions is reduced. In this case, we must take the limit $X\to\infty$. Doing so, we find analytically that a one-moment approximation is sufficient to describe all transport coefficients in the plane perpendicular to $\boldsymbol{B}$,

$$f_\perp = 0.2992, \quad r_H = 1, \quad a_\perp = 0, \quad N^* = 0, \quad L_\perp = 0, \quad L^* = 0. \tag{37}$$

The inclusion of higher-order moments does not affect these results. In the presence of a magnetic field, electrons in a low-density plasma undergo cyclotron motion which destroys all ee and higher-order ei correlations.

We now go beyond the low-density limit. In Fig. 1, isotherms for each transport coefficient are plotted in dependence of the coupling parameter. The FFCFs are calculated numerically applying a screened Coulomb potential within the Born approximation for both ei and ee interactions. Note again the decrease in the coefficients when ee interactions are included and the further decrease when the effect of the magnetic field is taken into account. At $\Gamma=0.01$, the low-density limits given in Eq. (37) are reached, except for the electrical conductivity where a convergence of 6% is reached. It appears to converge more slowly toward the limit of 0.2992.

Moving toward stronger correlated systems, the collision frequency in the system increases and the cyclotron motion is interrupted. As this occurs, the influence of the magnetic field on the transport coefficients diminishes and the results obtained in the small field limit ($X\to 0$) become relevant. Notably, both the ee and higher-order ei correlations become important once the cyclotron motion is disturbed. The transition between the results obtained in the limits $X\to\infty$ and $X\to 0$ occurs at higher density (larger Γ for a fixed T) as the magnetic field increases.

At solid densities, the electrons become degenerate and the ee and higher-order ei correlations are destroyed. Therefore, for large values of Γ, a one-moment approximation is again expected to be sufficient to describe all transport properties in the system. While this is clearly evident in the Hall coefficient, the return to the one-moment results does not fully occur for the other coefficients. However, LRT as applied here within perturbation theory is only valid up to about $\Gamma\approx 2$ (see Morozov *et al.* [27]). For strongly correlated systems, effects such as the structure factor have to be taken into account.

Additionally, in Fig. 2, the electrical conductivity is shown versus the free-electron density at a temperature of 10^4 K and assuming different magnetic field strengths. The transition between the low-density ($X\to\infty$) and the small field ($X\to 0$) limits can again be clearly seen to occur at larger densities for stronger magnetic fields. The densities at which this occurs, however, are far below those usually achieved in experiments.

Finally, in Fig. 3, the reduced electrical conductivity f is shown versus the ratio $\omega_e\tau$. This is to allow comparison to the results of Brown and Haines [10]. In Fig. 1, they show the resistivity $\alpha_\perp$, which is the inverse of the conductivity $\sigma_\perp$. Qualitatively, the results obtained here show very similar

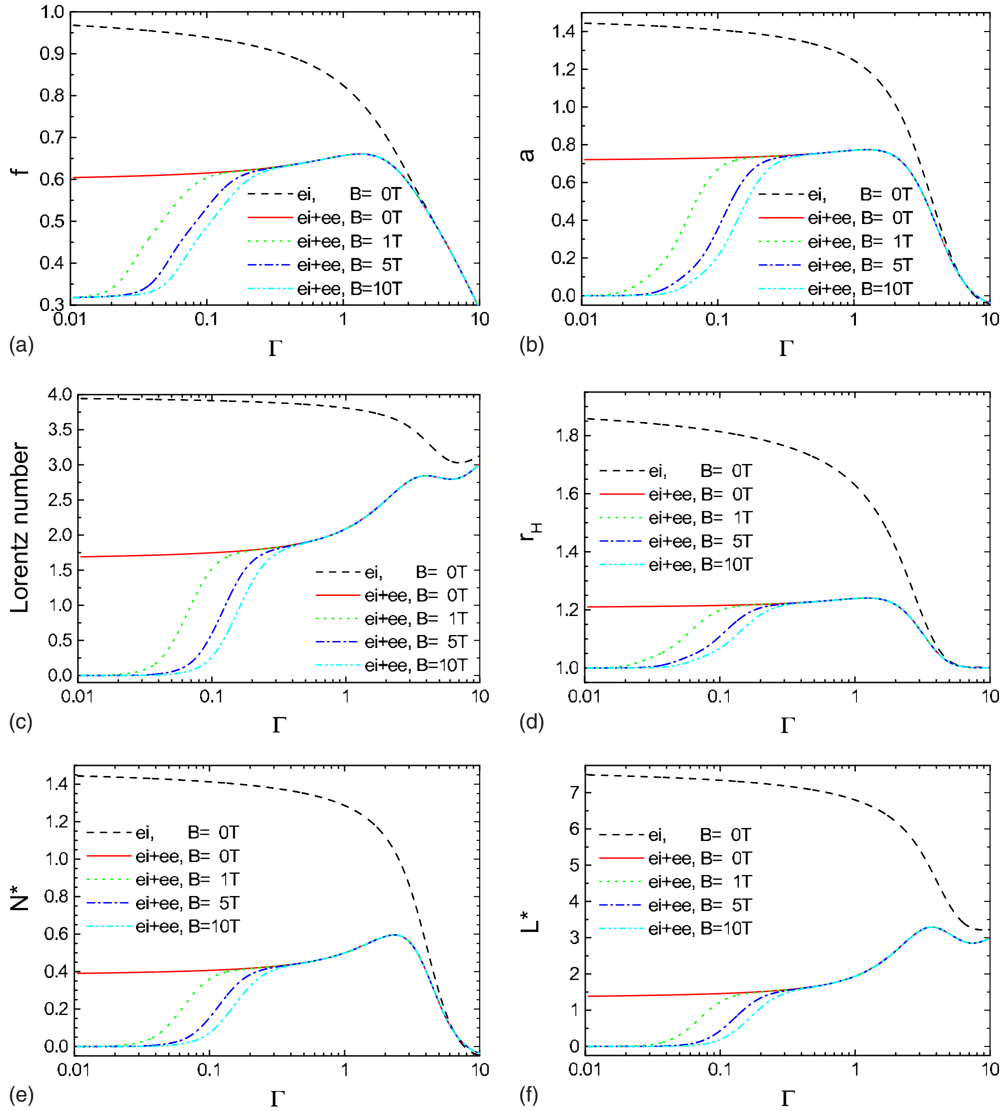

FIG. 1. (Color online) Dimensionless transport coefficients within five-moment approximations using screened potential in Born approximation as function of nonideality parameter Γ for hydrogen (ei+ee) and Lorentz plasma (ei) at 10^4 K and small magnetic field limit at different magnetic field strengths. Limits $\Gamma \to 0$ given in Tables I and II for $B=0$ and Eq. (37) for $B>0$.

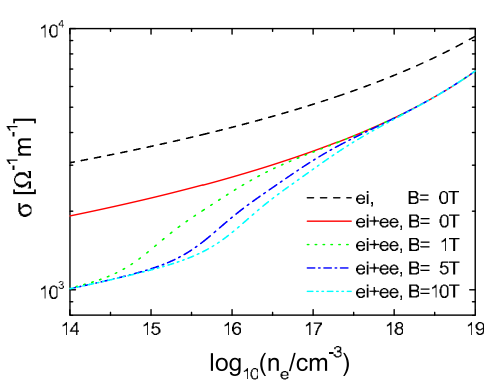

FIG. 2. (Color online) Electrical conductivity in five-moment approximation using screened potential in Born approximation for fully ionized plasma as function of free-electron density at 10^4 K and different magnetic field strengths.

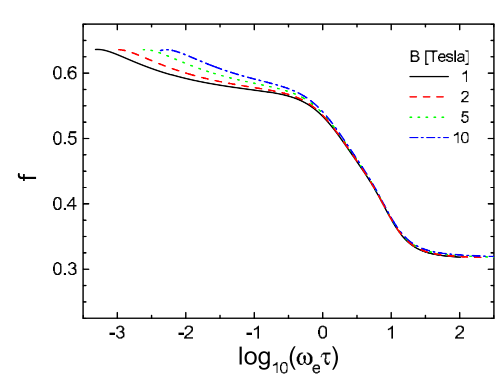

FIG. 3. (Color online) Reduced electrical conductivity in five-moment approximation using screened potential in Born approximation for fully ionized plasma as function of the ratio $\omega_e \tau$ at 10^4 K and different magnetic field strengths.

behavior. The conductivity of an unmagnetized plasma is reduced when ee interactions are considered, while the conductivity decreases and the ee interactions become negligible in a magnetized plasma.

V. PARTIALLY IONIZED PLASMAS

The temperature and free-electron density of experimentally produced plasmas vary depending on the materials to be heated and the method used to produce the plasma. Typically, a partially ionized plasma is created. In order to compare theoretical and experimental results, it is therefore necessary to take into account the ea interactions. This can be achieved consistently within LRT by including the ea contribution to the FFCF. Modifications to the RTA have been suggested in [28].

An additional difficulty when considering partially ionized plasmas is the determination of the plasma composition. In the case of heavier elements, atoms may become multiply ionized and the concentration of each ion species must be determined. Within this work, plasma compositions for a given temperature and density are calculated using the Fortran package COMPTRA04, which is described by Kuhlbrodt *et al.* [29]. At typical experimental densities $\rho < 1\ \mathrm{g/cm^3}$, compositions calculated using this package show only minor differences with those calculated using the comparable package SAHA4 developed by Gryaznov *et al.* [30,31], with most discrepancies attributable to the choice of thermodynamic model. For a detailed comparison of electrical conductivities of noble gases calculated within LRT using composition results from both models, see Adams *et al.* [11]. The FFCFs are calculated numerically applying a T-matrix approach to the ei and ee interactions, while using either experimental measurements (for noble gases) or a polarization potential (for hydrogen and noble gases) for the transport cross sections of ea interactions (see again Adams *et al.* [11]).

We present here theoretical results for the transport coefficients of argon, as this is a gas that lends itself well to experimental measurement [2]. It can also be considered to be prototypical for other elements. Experimental densities for argon are usually in the range of $(0.01-1)\ \mathrm{g/cm^3}$. Within this temperature and density range, we have $\Gamma > 0.1$. Therefore, having in mind the analysis of the fully ionized plasma (Fig. 1), we do not expect or predict any significant change in the electrical conductivity due to the presence of magnetic fields under 10 T. Proposed experiments using fields as large as 25 T would provide more significant deviation of the electrical conductivity from results expected in the absence of a magnetic field [36].

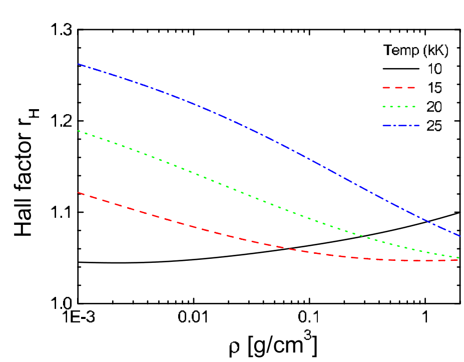

FIG. 4. (Color online) Hall factor of partially ionized Ar as function of density ρ for magnetic field of 5 T and different temperatures, five-moment approximation applying T-matrix approach for ei and ee interactions, and experimental data for the ea transport cross section [32–35].

Measurements of the Hall coefficient are considered for evidence of the magnetic field dependence of transport properties. In Fig. 4, the Hall factor of argon is shown against the density over a range of temperatures accessible in experiments. A first point to note is the possibility of Hall factors larger than the fully ionized limit of 1.20 being observed in partially ionized plasmas. The weak ea scattering mechanism has a greater impact on the current parallel to the electric field than on the Hall voltage, leading to a larger Hall coefficient. As temperature increases, the Hall factor remains at an enhanced value until larger densities due to the reduced influence of electron degeneracy at higher temperatures. Hall factors are predicted to be between 1 and 1.3, with only a gradual change between these values over a change of a few orders of magnitude in the density. Given the temperature and mass density of a plasma, a value for the electron density could be calculated using a composition program such as COMPTRA04 and an approximate value for r_H generated using the formulas described here. Alternatively, the electron density could be determined from experimental data using $n_e = -r_H/(eR_H)$ and compared to that calculated from plasma composition theories in order to validate them. Currently however, such an approach is limited by the uncertainty in experimental measurements which amount up to 30%–50% in the Hall coefficient [2]. This is larger than the discrepancies found between various composition programs [11]. Thus, discrimination of different plasma models remains a subject of further experimental work.

VI. CONCLUSIONS

In this work, an extension to linear response theory to include magnetic field effects has been presented. This allows a description of the Hall, Nernst, and Leduc-Righi coefficients, as well as a description of the influence of a magnetic field on the electrical and heat conductivities and the thermopower. The magnetic field has no influence on transport coefficients in direction parallel to it and previous results obtained neglecting magnetic fields are found to be applicable in this direction. Results for the Hall, Nernst, and Leduc-Righi coefficients have been obtained in the low-density limit with the inclusion of electron-electron interactions.

We considered arbitrarily strong magnetic fields. Significant decreases from previous small magnetic field approximations are predicted in the low-density limit for the transport coefficients in the plane perpendicular to the magnetic field. In particular, the dimensionless electrical conductivity is 0.2992 instead of 0.591 and the Hall factor is 1 rather than 1.93. The differences are attributable to a consistent treatment of the magnetic field effects at low density. Cyclotron motion disrupts electron-electron correlations, leading to all transport coefficients being described exactly by a one-moment approximation.

A distinct yet gradual transition between the results in the low-density limit and the small field approximation is predicted as the density increases and the influence of the magnetic field diminishes, creating a region with larger dimensionless transport coefficients. As degeneracy begins to dominate, electron-electron correlations are less relevant and the transport coefficients can again be accurately described by a one-moment approximation. Within temperature and density regions of experimental interest, the influence of the magnetic field on the electrical conductivity is significantly smaller than current experimental errors for magnetic fields less than around 10 T. A procedure for determining the ionization degree via the Hall coefficient may be viable in the future with improved accuracy of experimental measurements.

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